List of Symbols, Notations and Data

\( B(n, p) \): Binomial distribution with \( n \) trials and success probability \( p; \ n \in \{1, 2, \ldots\} \) and \( p \in (0, 1) \)

\( U(a, b) \): Uniform distribution on the interval \((a, b), -\infty < a < b < \infty\)

\( N(\mu, \sigma^2) \): Normal distribution with mean \( \mu \) and variance \( \sigma^2 \), \( \mu \in (-\infty, \infty), \sigma > 0 \)

\( P(A) \): Probability of the event \( A \)

\( \text{Poisson}(\lambda) \): Poisson distribution with mean \( \lambda, \lambda > 0 \)

\( E(X) \): Expected value (mean) of the random variable \( X \)

If \( Z \sim N(0, 1) \), then \( P(Z \leq 1.96) = 0.975 \) and \( P(Z \leq 0.54) = 0.7054 \)

\( Z \): Set of integers

\( \mathbb{Q} \): Set of rational numbers

\( \mathbb{R} \): Set of real numbers

\( \mathbb{C} \): Set of complex numbers

\( \mathbb{Z}_n \): The cyclic group of order \( n \)

\( \mathbb{F}[x] \): Polynomial ring over the field \( \mathbb{F} \)

\( C[0, 1] \): Set of all real valued continuous functions on the interval \([0, 1]\)

\( C^1[0, 1] \): Set of all real valued continuously differentiable functions on the interval \([0, 1]\)

\( \ell^2 \): Normed space of all square-summable real sequences

\( L^2[0, 1] \): Space of all square-Lebesgue integrable real valued functions on the interval \([0, 1]\)

\( (C[0, 1], \| \|_2) \): The space \( C[0, 1] \) with \( \| f \|_2 = \left( \int_0^1 f(x)^2 \ dx \right)^{1/2} \)

\( (C[0, 1], \| \|_\infty) \): The space \( C[0, 1] \) with \( \| f \|_\infty = \sup \{ |f(x)| : x \in [0, 1] \} \)

\( V^\perp \): The orthogonal complement of \( V \) in an inner product space

\( \mathbb{R}^n \): \( n \)-dimensional Euclidean space

Usual metric \( d \) on \( \mathbb{R}^n \) is given by \( d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} \)

\( I_n \): The \( n \times n \) identity matrix (\( I \): the identity matrix when order is NOT specified)

\( o(g) \): The order of the element \( g \) of a group
Q. 1 – Q. 25 carry one mark each.

Q.1 Let \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) be a linear map defined by
\[
T(x, y, z, w) = (x + z, 2x + y + 3z, 2y + 2z, w).
\]
Then the rank of \( T \) is equal to ________

Q.2 Let \( M \) be a \( 3 \times 3 \) matrix and suppose that 1, 2 and 3 are the eigenvalues of \( M \). If
\[
M^{-1} = \frac{M^2}{\alpha} - M + \frac{11}{\alpha} I_3
\]
for some scalar \( \alpha \neq 0 \), then \( \alpha \) is equal to ________

Q.3 Let \( M \) be a \( 3 \times 3 \) singular matrix and suppose that 2 and 3 are eigenvalues of \( M \). Then the number of linearly independent eigenvectors of \( M^3 + 2M + I_3 \) is equal to ________

Q.4 Let \( M \) be a \( 3 \times 3 \) matrix such that
\[
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -3 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
and suppose that
\[
M^3 \begin{pmatrix}
1/2 \\
1/2 \\
0
\end{pmatrix} = \begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
\]
for some \( \alpha, \beta, \gamma \in \mathbb{R} \). Then \( |\alpha| \) is equal to ______

Q.5 Let \( f : [0, \infty) \rightarrow \mathbb{R} \) be defined by
\[
f(x) = \int_0^x \sin^2(t^2) \, dt.
\]
Then the function \( f \) is
(A) uniformly continuous on \([0, 1)\) but NOT on \((0, \infty)\)
(B) uniformly continuous on \((0, \infty)\) but NOT on \([0, 1)\)
(C) uniformly continuous on both \([0, 1)\) and \((0, \infty)\)
(D) neither uniformly continuous on \([0, 1)\) nor uniformly continuous on \((0, \infty)\)

Q.6 Consider the power series \( \sum_{n=0}^{\infty} a_n z^n \), where
\[
a_n = \begin{cases}
\frac{1}{3^n} & \text{if } n \text{ is even} \\
\frac{1}{5^n} & \text{if } n \text{ is odd}
\end{cases}
\]
The radius of convergence of the series is equal to ________

Q.7 Let \( C = \{ z \in \mathbb{C} : |z - i| = 2 \} \). Then
\[
\frac{1}{2 \pi i} \oint_C \frac{z^2 - 4}{z^2 + 4} \, dz
\]
is equal to ________

Q.8 Let \( X \sim B\left(5, \frac{1}{2}\right) \) and \( Y \sim U(0,1) \). Then
\[
\frac{P(X+Y \leq 2)}{P(X+Y \geq 5)}
\]
is equal to ________
Q.9  Let the random variable $X$ have the distribution function

$$F(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{x}{2} & \text{if } 0 \leq x < 1 \\
\frac{3}{5} & \text{if } 1 \leq x < 2 \\
\frac{1}{2} + \frac{x}{8} & \text{if } 2 \leq x < 3 \\
1 & \text{if } x \geq 3.
\end{cases}$$

Then $P(2 \leq X < 4)$ is equal to _________.

Q.10  Let $X$ be a random variable having the distribution function

$$F(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{1}{4} & \text{if } 0 \leq x < 1 \\
\frac{1}{3} & \text{if } 1 \leq x < 2 \\
\frac{1}{2} & \text{if } 2 \leq x < \frac{11}{3} \\
1 & \text{if } x \geq \frac{11}{3}.
\end{cases}$$

Then $E(X)$ is equal to _________.

Q.11  In an experiment, a fair die is rolled until two sixes are obtained in succession. The probability that the experiment will end in the fifth trial is equal to

(A) $\frac{125}{6^5}$  
(B) $\frac{150}{6^5}$  
(C) $\frac{175}{6^5}$  
(D) $\frac{200}{6^5}$

Q.12  Let $x_1 = 2.2$, $x_2 = 4.3$, $x_3 = 3.1$, $x_4 = 4.5$, $x_5 = 1.1$ and $x_6 = 5.7$ be the observed values of a random sample of size 6 from a $U(\theta - 1, \theta + 4)$ distribution, where $\theta \in (0, \infty)$ is unknown. Then a maximum likelihood estimate of $\theta$ is equal to

(A) 1.8  
(B) 2.3  
(C) 3.1  
(D) 3.6

Q.13  Let $\Omega = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$ with boundary $\partial \Omega$. If $u(x, y)$ is the solution of the Dirichlet problem

$$u_{xx} + u_{yy} = 0 \quad \text{in } \Omega \quad u(x, y) = 1 - 2y^2 \quad \text{on } \partial \Omega,$$

then $u\left(\frac{1}{2}, 0\right)$ is equal to

(A) $-1$  
(B) $-\frac{1}{4}$  
(C) $\frac{1}{4}$  
(D) 1
Q.14 Let \( c \in \mathbb{Z}_3 \) be such that \( \frac{\mathbb{Z}_3[X]}{(x^3 + cx + 1)} \) is a field. Then \( c \) is equal to _________

Q.15 Let \( V = C^1[0,1], X = (C[0,1], \| \cdot \|_\infty) \) and \( Y = (C[0,1], \| \cdot \|_2) \). Then \( V \) is

(A) dense in \( X \) but NOT in \( Y \)
(B) dense in \( Y \) but NOT in \( X \)
(C) dense in both \( X \) and \( Y \)
(D) neither dense in \( X \) nor dense in \( Y \)

Q.16 Let \( T : (C[0,1], \| \cdot \|_\infty) \to \mathbb{R} \) be defined by \( T(f) = \int_0^1 2xf(x) \, dx \) for all \( f \in C[0,1] \). Then \( \| T \| \) is equal to _________

Q.17 Let \( \tau_1 \) be the usual topology on \( \mathbb{R} \). Let \( \tau_2 \) be the topology on \( \mathbb{R} \) generated by \( \mathcal{B} = \{(a, b) \subset \mathbb{R} : -\infty < a < b < \infty \} \). Then the set \( \{ x \in \mathbb{R} : 4 \sin^2 x \leq 1 \} \cup \{ \frac{\pi}{2} \} \) is

(A) closed in \((\mathbb{R}, \tau_1)\) but NOT in \((\mathbb{R}, \tau_2)\)
(B) closed in \((\mathbb{R}, \tau_2)\) but NOT in \((\mathbb{R}, \tau_1)\)
(C) closed in both \((\mathbb{R}, \tau_1)\) and \((\mathbb{R}, \tau_2)\)
(D) neither closed in \((\mathbb{R}, \tau_1)\) nor closed in \((\mathbb{R}, \tau_2)\)

Q.18 Let \( X \) be a connected topological space such that there exists a non-constant continuous function \( f : X \to \mathbb{R} \), where \( \mathbb{R} \) is equipped with the usual topology. Let \( f(X) = \{ f(x) : x \in X \} \). Then

(A) \( X \) is countable but \( f(X) \) is uncountable
(B) \( f(X) \) is countable but \( X \) is uncountable
(C) both \( f(X) \) and \( X \) are countable
(D) both \( f(X) \) and \( X \) are uncountable

Q.19 Let \( d_1 \) and \( d_2 \) denote the usual metric and the discrete metric on \( \mathbb{R} \), respectively. Let \( f : (\mathbb{R}, d_1) \to (\mathbb{R}, d_2) \) be defined by \( f(x) = x \), \( x \in \mathbb{R} \). Then

(A) \( f \) is continuous but \( f^{-1} \) is NOT continuous
(B) \( f^{-1} \) is continuous but \( f \) is NOT continuous
(C) both \( f \) and \( f^{-1} \) are continuous
(D) neither \( f \) nor \( f^{-1} \) is continuous

Q.20 If the trapezoidal rule with single interval \([0,1]\) is exact for approximating the integral \( \int_0^1 (x^3 - cx^2) \, dx \), then the value of \( c \) is equal to ________

Q.21 Suppose that the Newton-Raphson method is applied to the equation \( 2x^2 + 1 - e^{x^2} = 0 \) with an initial approximation \( x_0 \) sufficiently close to zero. Then, for the root \( x = 0 \), the order of convergence of the method is equal to ________
Q.22 The minimum possible order of a homogeneous linear ordinary differential equation with real constant coefficients having \(x^2 \sin(x)\) as a solution is equal to ________

Q.23 The Lagrangian of a system in terms of polar coordinates \((r, \theta)\) is given by

\[
L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m (\dot{\theta}^2 + r^2 \dot{\theta}^2) - m g r (1 - \cos(\theta)),
\]

where \(m\) is the mass, \(g\) is the acceleration due to gravity and \(\dot{s}\) denotes the derivative of \(s\) with respect to time. Then the equations of motion are

(A) \(2 \ddot{r} = r \dot{\theta}^2 - g (1 - \cos(\theta))\), \(\frac{d}{dt}(r^2 \dot{\theta}) = -g r \sin(\theta)\)

(B) \(2 \ddot{r} = r \dot{\theta}^2 + g (1 - \cos(\theta))\), \(\frac{d}{dt}(r^2 \dot{\theta}) = -g r \sin(\theta)\)

(C) \(2 \ddot{r} = r \dot{\theta}^2 - g (1 - \cos(\theta))\), \(\frac{d}{dt}(r^2 \dot{\theta}) = g r \sin(\theta)\)

(D) \(2 \ddot{r} = r \dot{\theta}^2 + g (1 - \cos(\theta))\), \(\frac{d}{dt}(r^2 \dot{\theta}) = g r \sin(\theta)\)

Q.24 If \(y(x)\) satisfies the initial value problem

\[(x^2 + y)dx = x \, dy, \quad y(1) = 2,\]

then \(y(2)\) is equal to ________

Q.25 It is known that Bessel functions \(J_n(x)\), for \(n \geq 0\), satisfy the identity

\[e^{\frac{x}{2}} (t - \frac{1}{t}) = J_0(x) + \sum_{n=1}^{\infty} J_n(x) \left( t^n + \frac{(-1)^n}{t^n} \right)\]

for all \(t > 0\) and \(x \in \mathbb{R}\). The value of \(J_0 \left( \frac{\pi}{3} \right) + 2 \sum_{n=1}^{\infty} J_{2n} \left( \frac{\pi}{3} \right)\) is equal to ________

Q.26 – Q.55 carry two marks each.

Q.26 Let \(X\) and \(Y\) be two random variables having the joint probability density function

\[f(x, y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}\]

Then the conditional probability \(P \left( X \leq \frac{2}{3} \mid Y = \frac{3}{4} \right)\) is equal to

(A) \(\frac{5}{9}\) \quad (B) \(\frac{2}{3}\) \quad (C) \(\frac{7}{9}\) \quad (D) \(\frac{8}{9}\)

Q.27 Let \(\Omega = (0,1]\) be the sample space and let \(P(\cdot)\) be a probability function defined by

\[P((0, x]) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x < \frac{1}{2} \\ x & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}\]

Then \(P \left( \left[ \frac{1}{2}, 1 \right) \right)\) is equal to __________
Q.28 Let $X_1, X_2$ and $X_3$ be independent and identically distributed random variables with $E(X_1) = 0$ and $E(X_1^2) = \frac{15}{4}$. If $\psi: (0, \infty) \to (0, \infty)$ is defined through the conditional expectation $\psi(t) = E(X_1^2 | X_1 + X_2 + X_3^2 = t), \ t > 0$, then $E(\psi((X_1 + X_2)^2))$ is equal to ________

Q.29 Let $X \sim \text{Poisson}(\lambda)$, where $\lambda > 0$ is unknown. If $\delta(X)$ is the unbiased estimator of $g(\lambda) = e^{-\lambda}(3\lambda^2 + 2\lambda + 1)$, then $\sum_{k=0}^{\infty} \delta(k)$ is equal to ________

Q.30 Let $X_1, ..., X_n$ be a random sample from $N(\mu, 1)$ distribution, where $\mu \in \{0, \frac{1}{2}\}$. For testing the null hypothesis $H_0: \mu = 0$ against the alternative hypothesis $H_1: \mu = \frac{1}{2}$ consider the critical region $R = \{(x_1, x_2, ..., x_n) : \sum_{i=1}^{n} x_i > c \}$, where $c$ is some real constant. If the critical region $R$ has size 0.025 and power 0.7054, then the value of the sample size $n$ is equal to ________

Q.31 Let $X$ and $Y$ be independently distributed central chi-squared random variables with degrees of freedom $m (\geq 3)$ and $n (\geq 3)$, respectively. If $E\left(\frac{X}{Y}\right) = 3$ and $m + n = 14$, then $E\left(\frac{X}{Y}\right)$ is equal to ________

(A) $\frac{2}{7}$  
(B) $\frac{3}{7}$  
(C) $\frac{4}{7}$  
(D) $\frac{5}{7}$

Q.32 Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with $P(X_1 = 1) = \frac{1}{4}$ and $P(X_1 = 2) = \frac{3}{4}$. If $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$, for $n = 1, 2, \ldots$, then $\lim_{n \to \infty} P(\bar{X}_n \leq 1.8)$ is equal to ________

Q.33 Let $u(x, y) = 2f(y)\cos(x - 2y), \ (x, y) \in \mathbb{R}^2$, be a solution of the initial value problem

$$2u_x + u_y = u$$
$$u(x, 0) = \cos(x).$$

Then $f(1)$ is equal to ________

(A) $\frac{1}{2}$  
(B) $\frac{e}{2}$  
(C) $e$  
(D) $\frac{3e}{2}$

Q.34 Let $u(x, t)$, $x \in \mathbb{R}$, $t \geq 0$, be the solution of the initial value problem

$$u_{tt} = u_{xx}$$
$$u(x, 0) = x$$
$$u_t(x, 0) = 1.$$ 

Then $u(2, 2)$ is equal to ________
Q.35 Let \( W = \text{Span} \left\{ \frac{1}{\sqrt{2}} (0,0,1,1), \frac{1}{\sqrt{2}} (1,-1,0,0) \right\} \) be a subspace of the Euclidean space \( \mathbb{R}^4 \). Then the square of the distance from the point \((1,1,1,1)\) to the subspace \( W \) is equal to ________.

Q.36 Let \( T: \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) be a linear map such that the null space of \( T \) is \( \{ (x,y,z,w) \in \mathbb{R}^4 : x + y + z + w = 0 \} \) and the rank of \( (T - 4 I_4) \) is 3. If the minimal polynomial of \( T \) is \( x(x - 4)^\alpha \), then \( \alpha \) is equal to ________.

Q.37 Let \( M \) be an invertible Hermitian matrix and let \( x, y \in \mathbb{R} \) be such that \( x^2 < 4y \). Then

(A) both \( M^2 + xM + y I \) and \( M^2 - xM + y I \) are singular
(B) \( M^2 + xM + y I \) is singular but \( M^2 - xM + y I \) is non-singular
(C) \( M^2 + xM + y I \) is non-singular but \( M^2 - xM + y I \) is singular
(D) both \( M^2 + xM + y I \) and \( M^2 - xM + y I \) are non-singular

Q.38 Let \( G = \{ e, x, x^2, x^3, y, xy, x^2 y, x^3 y \} \) with \( o(x) = 4, o(y) = 2 \) and \( xy = yx^3 \). Then the number of elements in the center of the group \( G \) is equal to

(A) 1  (B) 2  (C) 4  (D) 8

Q.39 The number of ring homomorphisms from \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) to \( \mathbb{Z}_4 \) is equal to ________.

Q.40 Let \( p(x) = 9x^5 + 10x^3 + 5x + 15 \) and \( q(x) = x^3 - x^2 - x - 2 \) be two polynomials in \( \mathbb{Q}[x] \). Then, over \( \mathbb{Q} \),

(A) \( p(x) \) and \( q(x) \) are both irreducible
(B) \( p(x) \) is reducible but \( q(x) \) is irreducible
(C) \( p(x) \) is irreducible but \( q(x) \) is reducible
(D) \( p(x) \) and \( q(x) \) are both reducible

Q.41 Consider the linear programming problem

\[
\begin{align*}
\text{Maximize} & \quad 3x + 9y, \\
\text{subject to} & \quad 2y - x \leq 2 \\
& \quad 3y - x \geq 0 \\
& \quad 2x + 3y \leq 10 \\
& \quad x, y \geq 0.
\end{align*}
\]

Then the maximum value of the objective function is equal to ________.

Q.42 Let \( S = \{ (x, \sin \frac{1}{x}) : 0 < x \leq 1 \} \) and \( T = S \cup \{(0,0)\} \). Under the usual metric on \( \mathbb{R}^2 \),

(A) \( S \) is closed but \( T \) is NOT closed
(B) \( T \) is closed but \( S \) is NOT closed
(C) both \( S \) and \( T \) are closed
(D) neither \( S \) nor \( T \) is closed
Q.43 Let \( H = \left\{ (x_n) \in l^2 : \sum_{n=1}^{\infty} \frac{x_n}{n} = 1 \right\} \). Then \( H \)
(A) is bounded (B) is closed (C) is a subspace (D) has an interior point

Q.44 Let \( V \) be a closed subspace of \( L^2[0, 1] \) and let \( f, g \in L^2[0, 1] \) be given by \( f(x) = x \) and \( g(x) = x^2 \). If \( V^\perp = \text{Span} \left\{ f \right\} \) and \( P \) is the orthogonal projection of \( g \) on \( V \), then \( (g - Pg)(x), \ x \in [0, 1] \), is
(A) \( \frac{3}{4}x \) (B) \( \frac{1}{4}x \) (C) \( \frac{3}{4}x^2 \) (D) \( \frac{1}{4}x^2 \)

Q.45 Let \( p(x) \) be the polynomial of degree at most 3 that passes through the points \((-2, 12), (-1, 1), (0, 2) \) and \( (2, -8) \). Then the coefficient of \( x^3 \) in \( p(x) \) is equal to ________

Q.46 If, for some \( \alpha, \beta \in \mathbb{R} \), the integration formula
\[
\int_0^2 p(x) \, dx = p(\alpha) + p(\beta)
\]
holds for all polynomials \( p(x) \) of degree at most 3, then the value of \( 3(\alpha - \beta)^2 \) is equal to _____

Q.47 Let \( y(t) \) be a continuous function on \([0, \infty) \) whose Laplace transform exists. If \( y(t) \) satisfies
\[
\int_0^t (1 - \cos(t - \tau)) \, y(\tau) \, d\tau = t^4,
\]
then \( y(1) \) is equal to ________

Q.48 Consider the initial value problem
\[
x^2 y'' - 6y = 0, \quad y(1) = \alpha, \ y'(1) = 6.
\]
If \( y(x) \to 0 \) as \( x \to 0^+ \), then \( \alpha \) is equal to ________

Q.49 Define \( f_1, f_2 : [0, 1] \to \mathbb{R} \) by
\[
f_1(x) = \sum_{n=1}^{\infty} \frac{x \sin(n^2 x)}{n^2} \quad \text{and} \quad f_2(x) = \sum_{n=1}^{\infty} x^2(1 - x^2)^{n-1}.
\]
Then
(A) \( f_1 \) is continuous but \( f_2 \) is NOT continuous
(B) \( f_2 \) is continuous but \( f_1 \) is NOT continuous
(C) both \( f_1 \) and \( f_2 \) are continuous
(D) neither \( f_1 \) nor \( f_2 \) is continuous

Q.50 Consider the unit sphere \( S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \) and the unit normal vector \( \hat{n} = (x, y, z) \) at each point \((x, y, z)\) on \( S \). The value of the surface integral
\[
\int_S \left\{ \left(\frac{2x}{\pi} + \sin(y^2)\right) x + \left(\frac{e^x}{\pi} - \frac{y}{\pi}\right) y + \left(\frac{2z}{\pi} + \sin^2 y\right) z \right\} \, d\sigma
\]
is equal to ________
Q.51 Let \( D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 1000, \ 1 \leq y \leq 1000\} \). Define

\[
f(x, y) = \frac{xy}{2} + \frac{500}{x} + \frac{500}{y}.
\]

Then the minimum value of \( f \) on \( D \) is equal to ________

Q.52 Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). Then there exists a non-constant analytic function \( f \) on \( \mathbb{D} \) such that for all \( n = 2, 3, 4, ... \)

(A) \( f \left( \frac{\sqrt{-1}}{n} \right) = 0 \)  \hspace{1cm} (B) \( f \left( \frac{1}{n} \right) = 0 \)

(C) \( f \left( 1 - \frac{1}{n} \right) = 0 \)  \hspace{1cm} (D) \( f \left( \frac{1}{2} - \frac{1}{n} \right) = 0 \)

Q.53 Let \( \sum_{n=-\infty}^{\infty} a_n z^n \) be the Laurent series expansion of \( f(z) = \frac{1}{2 z^2 - 13 z + 15} \) in the annulus \( \frac{3}{2} < |z| < 5 \). Then \( \frac{a_1}{a_2} \) is equal to _________

Q.54 The value of \( \frac{i}{4-\pi} \int_{|z|=4} \frac{dz}{z \cos(z)} \) is equal to _________

Q.55 Suppose that among all continuously differentiable functions \( y(x), \ x \in \mathbb{R}, \) with \( y(0) = 0 \) and \( y(1) = \frac{1}{2} \), the function \( y_0(x) \) minimizes the functional

\[
\int_0^1 \left( e^{-(y'-x)} + (1 + y)y' \right) dx.
\]

Then \( y_0 \left( \frac{1}{2} \right) \) is equal to

(A) \( 0 \)  \hspace{1cm} (B) \( \frac{1}{8} \)  \hspace{1cm} (C) \( \frac{1}{4} \)  \hspace{1cm} (D) \( \frac{1}{2} \)

END OF THE QUESTION PAPER