### Special Instructions / Useful Data

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{R} )</td>
<td>Set of all real numbers</td>
</tr>
<tr>
<td>( \mathbb{R}^n )</td>
<td>( {(x_1, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, \ldots, n} )</td>
</tr>
<tr>
<td>( P(A) )</td>
<td>Probability of an event ( A )</td>
</tr>
<tr>
<td>i.i.d.</td>
<td>Independently and identically distributed</td>
</tr>
<tr>
<td>( \text{Bin}(n, p) )</td>
<td>Binomial distribution with parameters ( n ) and ( p )</td>
</tr>
<tr>
<td>( \text{Poisson}(\theta) )</td>
<td>Poisson distribution with mean ( \theta )</td>
</tr>
<tr>
<td>( \text{N}(\mu, \sigma^2) )</td>
<td>Normal distribution with mean ( \mu ) and variance ( \sigma^2 )</td>
</tr>
<tr>
<td>( \text{Exp}(\lambda) )</td>
<td>The exponential distribution with probability density function</td>
</tr>
<tr>
<td>( f(x</td>
<td>\lambda) = \begin{cases} \lambda e^{-\lambda x}, &amp; x &gt; 0, \ 0, &amp; \text{otherwise} \end{cases}, \ \lambda &gt; 0 )</td>
</tr>
<tr>
<td>( t_n )</td>
<td>Student’s ( t ) distribution with ( n ) degrees of freedom</td>
</tr>
<tr>
<td>( \chi_n^2 )</td>
<td>Chi-square distribution with ( n ) degrees of freedom</td>
</tr>
<tr>
<td>( \chi^2_{n, \alpha} )</td>
<td>A constant such that ( P(W &gt; \chi^2_{n, \alpha}) = \alpha ), where ( W ) has ( \chi_n^2 ) distribution</td>
</tr>
<tr>
<td>( \Phi(x) )</td>
<td>Cumulative distribution function of ( \text{N}(0,1) )</td>
</tr>
<tr>
<td>( \phi(x) )</td>
<td>Probability density function of ( \text{N}(0,1) )</td>
</tr>
<tr>
<td>( A^c )</td>
<td>Complement of an event ( A )</td>
</tr>
<tr>
<td>( E(X) )</td>
<td>Expectation of a random variable ( X )</td>
</tr>
<tr>
<td>( \text{Var}(X) )</td>
<td>Variance of a random variable ( X )</td>
</tr>
<tr>
<td>( B(m, n) )</td>
<td>( \int_0^1 x^{m-1} (1-x)^{n-1} , dx, \ \ m &gt; 0, \ n &gt; 0 )</td>
</tr>
<tr>
<td>([x])</td>
<td>The greatest integer less than or equal to real number ( x )</td>
</tr>
<tr>
<td>( f' )</td>
<td>Derivative of function ( f )</td>
</tr>
</tbody>
</table>

\[ \Phi(0.25) = 0.5987, \Phi(0.5) = 0.6915, \Phi(0.625) = 0.7341, \Phi(0.71) = 0.7612, \]
\[ \Phi(1) = 0.8413, \Phi(1.125) = 0.8697, \Phi(2) = 0.9772 \]
SECTION – A
MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q.10 carry one mark each.

Q.1 Let
\[ P = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & -2 & 1 & 1 \\ 1 & 2 & -3 & -7 \\ 1 & 2 & -2 & -4 \end{bmatrix}. \]
Then rank of \( P \) equals
(A) 4
(B) 3
(C) 2
(D) 1

Q.2 Let \( \alpha, \beta, \gamma \) be real numbers such that \( \beta \neq 0 \) and \( \gamma \neq 0 \). Suppose
\[ P = \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}, \]
and \( P^{-1} = P \). Then
(A) \( \alpha = 0 \) and \( \beta \gamma = 1 \)
(B) \( \alpha \neq 0 \) and \( \beta \gamma = 1 \)
(C) \( \alpha = 0 \) and \( \beta \gamma = 2 \)
(D) \( \alpha = 0 \) and \( \beta \gamma = -1 \)

Q.3 Let \( m > 1 \). The volume of the solid generated by revolving the region between the \( y \)-axis and the curve \( xy = 4, 1 \leq y \leq m \), about the \( y \)-axis is \( 15 \pi \). The value of \( m \) is
(A) 14
(B) 15
(C) 16
(D) 17

Q.4 Consider the region \( S \) enclosed by the surface \( z = y^2 \) and the planes \( z = 1, x = 0, x = 1, y = -1 \) and \( y = 1 \). The volume of \( S \) is
(A) \( \frac{1}{3} \)
(B) \( \frac{2}{3} \)
(C) 1
(D) \( \frac{4}{3} \)
Q.5 Let \( X \) be a discrete random variable with the moment generating function
\[
M_X(t) = e^{0.5(t^2-1)}, \ t \in \mathbb{R}.
\]
Then \( P(X \leq 1) \) equals
(A) \( e^{-\frac{1}{2}} \)  
(B) \( \frac{3}{2} e^{-\frac{1}{2}} \)  
(C) \( \frac{1}{2} e^{-\frac{1}{2}} \)  
(D) \( e^{-(\frac{1}{2})^2} \)

Q.6 Let \( E \) and \( F \) be two independent events with
\[
P(E|F) + P(F|E) = 1, \ P(E \cap F) = \frac{2}{9} \quad \text{and} \quad P(F) < P(E).
\]
Then \( P(E) \) equals
(A) \( \frac{1}{3} \)  
(B) \( \frac{1}{2} \)  
(C) \( \frac{2}{3} \)  
(D) \( \frac{3}{4} \)

Q.7 Let \( X \) be a continuous random variable with the probability density function
\[
f(x) = \frac{1}{(2+x^2)^{\frac{1}{2}}} , \ x \in \mathbb{R}.
\]
Then \( E(X^2) \)
(A) equals 0  
(B) equals 1  
(C) equals 2  
(D) does not exist

Q.8 The probability density function of a random variable \( X \) is given by
\[
f(x) = \begin{cases} \alpha x^{\alpha-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} , \ \alpha > 0.
\]
Then the distribution of the random variable \( Y = \log_x X^{-2} \) is
(A) \( \chi_2^2 \)  
(B) \( \frac{1}{2} \chi_2^2 \)  
(C) \( 2\chi_2^2 \)  
(D) \( \chi_1^2 \)

Q.9 Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. \( N(0,1) \) random variables. Then, as \( n \to \infty \), \( \frac{1}{n} \sum_{i=1}^{n} X_i^2 \) converges in probability to
(A) 0  
(B) 0.5  
(C) 1  
(D) 2
Q.10 Consider the simple linear regression model with \( n \) random observations \( Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad i = 1, \ldots, n \), \( n > 2 \). \( \beta_0 \) and \( \beta_1 \) are unknown parameters, \( x_1, \ldots, x_n \) are observed values of the regressor variable and \( \varepsilon_1, \ldots, \varepsilon_n \) are error random variables with \( E(\varepsilon_i) = 0, \ i = 1, \ldots, n \), and for \( i, j = 1, \ldots, n \),
\[
\text{Cov}(\varepsilon_i, \varepsilon_j) = \begin{cases} 
0, & \text{if } i \neq j, \\
\sigma^2, & \text{if } i = j.
\end{cases}
\]
For real constants \( a_1, \ldots, a_n \), if \( \sum_{i=1}^{n} a_i Y_i \) is an unbiased estimator of \( \beta_1 \), then

(A) \( \sum_{i=1}^{n} a_i = 0 \) and \( \sum_{i=1}^{n} a_i x_i = 0 \)

(B) \( \sum_{i=1}^{n} a_i = 0 \) and \( \sum_{i=1}^{n} a_i x_i = 1 \)

(C) \( \sum_{i=1}^{n} a_i = 1 \) and \( \sum_{i=1}^{n} a_i x_i = 0 \)

(D) \( \sum_{i=1}^{n} a_i = 1 \) and \( \sum_{i=1}^{n} a_i x_i = 1 \)

Q.11 – Q.30 carry two marks each.

Q.11 Let \((X, Y)\) have the joint probability density function
\[
f(x, y) = \begin{cases} 
\frac{1}{2} y^2 e^{-y}, & \text{if } 0 < y < x < \infty, \\
0, & \text{otherwise}.
\end{cases}
\]
Then \( P(Y < 1 \mid X = 3) \) equals

(A) \( \frac{1}{81} \)

(B) \( \frac{1}{27} \)

(C) \( \frac{1}{9} \)

(D) \( \frac{1}{3} \)

Q.12 Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. random variables having the probability density function
\[
f(x) = \begin{cases} 
\frac{1}{B(6, 4)} x^3 (1-x)^3, & 0 < x < 1, \\
0, & \text{otherwise}.
\end{cases}
\]
Let \( Y_i = \frac{X_i}{1-X_i} \) and \( U_n = \frac{1}{n} \sum_{i=1}^{n} Y_i \). If the distribution of \( \frac{\sqrt{n}(U_n - 2)}{\alpha} \) converges to \( N(0,1) \) as \( n \to \infty \), then a possible value of \( \alpha \) is

(A) \( \sqrt{7} \)

(B) \( \sqrt{5} \)

(C) \( \sqrt{3} \)

(D) 1
Q.13  Let $X_1, \ldots, X_n$ be a random sample from a population with the probability density function

$$f(x \mid \theta) = \begin{cases} 4e^{-4(x-\theta)}, & x > \theta, \\ 0, & \text{otherwise,} \end{cases} \quad \theta \in \mathbb{R}.$$ 

If $T_n = \min \{X_1, \ldots, X_n\}$, then

(A) $T_n$ is unbiased and consistent estimator of $\theta$

(B) $T_n$ is biased and consistent estimator of $\theta$

(C) $T_n$ is unbiased but NOT consistent estimator of $\theta$

(D) $T_n$ is NEITHER unbiased NOR consistent estimator of $\theta$

Q.14  Let $X_1, \ldots, X_n$ be i.i.d. random variables with the probability density function

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

If $X_{(n)} = \max \{X_1, \ldots, X_n\}$, then

$$\lim_{n \to \infty} P(X_{(n)} - \log n \leq 2)$$

equals

(A) $1 - e^{-2}$

(B) $e^{-e^{-0.5}}$

(C) $e^{-e^{-2}}$

(D) $e^{-e^2}$

Q.15  Let $X$ and $Y$ be two independent $N(0, 1)$ random variables. Then

$$P(0 < X^2 + Y^2 < 4)$$

equals

(A) $1 - e^{-2}$

(B) $1 - e^{-4}$

(C) $1 - e^{-1}$

(D) $e^{-2}$

Q.16  Let $X$ be a random variable with the cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{8}, & 0 \leq x < 2, \\ \frac{x^2}{16}, & 2 \leq x < 4, \\ 1, & x \geq 4. \end{cases}$$

Then $E(X)$ equals

(A) $\frac{12}{31}$

(B) $\frac{13}{12}$

(C) $\frac{31}{21}$

(D) $\frac{31}{12}$
Q.17 Let \( X_1, \ldots, X_n \) be a random sample from a population with the probability density function
\[
f(x) = \frac{1}{2\theta} e^{-|x|/\theta}, \ x \in \mathbb{R}, \ \theta > 0.
\]

For a suitable constant \( K \), the critical region of the most powerful test for testing \( H_0: \theta = 1 \) against \( H_1: \theta = 2 \) is of the form

(A) \( \sum_{i=1}^{n} |X_i| > K \)  
(B) \( \sum_{i=1}^{n} |X_i| < K \)  
(C) \( \sum_{i=1}^{n} \frac{1}{|X_i|} < K \)  
(D) \( \sum_{i=1}^{n} \frac{1}{|X_i|} > K \)

Q.18 Let \( X_1, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots, X_{n+m} \) \((n > 4, m > 4)\) be a random sample from \( N(\mu, \sigma^2) \); \( \mu \in \mathbb{R}, \sigma > 0 \). If \( \bar{X}_1 = \frac{1}{n} \sum_{i=1}^{n} X_i \) and \( \bar{X}_2 = \frac{1}{m-2} \sum_{i=n+1}^{n+m-2} X_i \), then the distribution of the random variable
\[
T = \frac{X_{n+m} - X_{n+m-1}}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X}_1)^2 + \sum_{i=n+1}^{n+m-2}(X_i - \bar{X}_2)^2}}
\]
is

(A) \( t_{n+m-2} \)  
(B) \( \sqrt{\frac{2}{n+m-1}} t_{n+m-1} \)  
(C) \( \sqrt{\frac{2}{n+m-4}} t_{n+m-4} \)  
(D) \( t_{n+m-4} \)

Q.19 Let \( X_1, \ldots, X_n \) \((n > 1)\) be a random sample from a \( \text{Poisson}(\theta) \) population, \( \theta > 0 \), and
\[
T = \sum_{i=1}^{n} X_i.
\]Then the uniformly minimum variance unbiased estimator of \( \theta^2 \) is

(A) \( \frac{T(T-1)}{n^2} \)  
(B) \( \frac{T(T-1)}{n(n-1)} \)  
(C) \( \frac{T(T-1)}{n(n+1)} \)  
(D) \( \frac{T^2}{n^2} \)
Q.20 Let $X$ be a random variable whose probability mass functions $f(x \mid H_0)$ (under the null hypothesis $H_0$) and $f(x \mid H_1)$ (under the alternative hypothesis $H_1$) are given by

<table>
<thead>
<tr>
<th>$X = x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x \mid H_0)$</td>
<td>0.4</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>$f(x \mid H_1)$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
</tr>
</tbody>
</table>

For testing the null hypothesis $H_0 : X \sim f(x \mid H_0)$ against the alternative hypothesis $H_1 : X \sim f(x \mid H_1)$, consider the test given by: Reject $H_0$ if $X > \frac{3}{2}$.

If $\alpha = \text{size of the test}$ and $\beta = \text{power of the test}$, then

(A) $\alpha = 0.3$ and $\beta = 0.3$

(B) $\alpha = 0.3$ and $\beta = 0.7$

(C) $\alpha = 0.7$ and $\beta = 0.3$

(D) $\alpha = 0.7$ and $\beta = 0.7$

Q.21 Let $X_1, \ldots, X_n$ be a random sample from a $N(2\theta, \theta^2)$ population, $\theta > 0$. A consistent estimator for $\theta$ is

(A) $\frac{1}{n} \sum_{i=1}^{n} X_i$

(B) $\left( \frac{5}{n^2} \sum_{i=1}^{n} X^2_i \right)^{1/2}$

(C) $\frac{1}{5n} \sum_{i=1}^{n} X^2_i$

(D) $\left( \frac{1}{5n} \sum_{i=1}^{n} X^2_i \right)^{1/2}$

Q.22 An institute purchases laptops from either vendor $V_1$ or vendor $V_2$ with equal probability. The lifetimes (in years) of laptops from vendor $V_1$ have a $U(0, 4)$ distribution, and the lifetimes (in years) of laptops from vendor $V_2$ have an $Exp(1/2)$ distribution. If a randomly selected laptop in the institute has lifetime more than two years, then the probability that it was supplied by vendor $V_2$ is

(A) $\frac{2}{2 + e}$

(B) $\frac{1}{1 + e}$

(C) $\frac{1}{1 + e^{-1}}$

(D) $\frac{2}{2 + e^{-1}}$
Q.23 Let \( y(x) \) be the solution to the differential equation

\[
x^4 \frac{dy}{dx} + 4x^3 y + \sin x = 0; \quad y(\pi) = 1, \quad x > 0.
\]

Then \( y\left(\frac{\pi}{2}\right) \) is

(A) \( \frac{10 \left(1 + \pi^4\right)}{\pi^4} \) 

(B) \( \frac{12 \left(1 + \pi^4\right)}{\pi^4} \)

(C) \( \frac{14 \left(1 + \pi^4\right)}{\pi^4} \) 

(D) \( \frac{16 \left(1 + \pi^4\right)}{\pi^4} \)

Q.24 Let \( a_n = e^{-2n} \sin n \) and \( b_n = e^{-n} n^2 \left(\sin n\right)^2 \) for \( n \geq 1 \). Then

(A) \( \sum_{n=1}^{\infty} a_n \) converges but \( \sum_{n=1}^{\infty} b_n \) does NOT converge

(B) \( \sum_{n=1}^{\infty} b_n \) converges but \( \sum_{n=1}^{\infty} a_n \) does NOT converge

(C) both \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) converge

(D) NEITHER \( \sum_{n=1}^{\infty} a_n \) NOR \( \sum_{n=1}^{\infty} b_n \) converges

Q.25 Let

\[
f(x) = \begin{cases} x \sin^2(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x \left(\sin x\right) \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}
\]

Then

(A) \( f \) is differentiable at 0 but \( g \) is NOT differentiable at 0

(B) \( g \) is differentiable at 0 but \( f \) is NOT differentiable at 0

(C) \( f \) and \( g \) are both differentiable at 0

(D) NEITHER \( f \) NOR \( g \) is differentiable at 0
Q.26 Let \( f : [0, 4] \rightarrow \mathbb{R} \) be a twice differentiable function. Further, let \( f(0) = 1 \), \( f(2) = 2 \) and \( f(4) = 3 \). Then

(A) there does NOT exist any \( x_i \in (0, 2) \) such that \( f'(x_i) = \frac{1}{2} \)
(B) there exist \( x_2 \in (0, 2) \) and \( x_3 \in (2, 4) \) such that \( f'(x_2) = f'(x_3) \)
(C) \( f''(x) > 0 \) for all \( x \in (0, 4) \)
(D) \( f''(x) < 0 \) for all \( x \in (0, 4) \)

Q.27 Let \( f(x, y) = x^2 - 400x y^2 \) for all \( (x, y) \in \mathbb{R}^2 \). Then \( f \) attains its

(A) local minimum at \((0, 0)\) but NOT at \((1, 1)\)
(B) local minimum at \((1, 1)\) but NOT at \((0, 0)\)
(C) local minimum both at \((0, 0)\) and \((1, 1)\)
(D) local minimum NEITHER at \((0, 0)\) NOR at \((1, 1)\)

Q.28 Let \( y(x) \) be the solution to the differential equation

\[
4 \frac{d^2 y}{dx^2} + 12 \frac{dy}{dx} + 9 y = 0, \quad y(0) = 1, \quad y'(0) = -4.
\]

Then \( y(1) \) equals

(A) \(-\frac{1}{2} e^{-3/2}\) \hspace{2cm} (B) \(-\frac{3}{2} e^{-3/2}\)
(C) \(-\frac{5}{2} e^{-3/2}\) \hspace{2cm} (D) \(-\frac{7}{2} e^{-3/2}\)

Q.29 Let \( g : [0, 2] \rightarrow \mathbb{R} \) be defined by

\[
g(x) = \int_0^x (x-t)e^t \, dt.
\]

The area between the curve \( y = g''(x) \) and the x-axis over the interval \([0, 2]\) is

(A) \(e^2 - 1\) \hspace{2cm} (B) \(2(e^2 - 1)\)
(C) \(4(e^2 - 1)\) \hspace{2cm} (D) \(8(e^2 - 1)\)
Q.30 Let $P$ be a $3 \times 3$ singular matrix such that $P \tilde{v} = \tilde{v}$ for a nonzero vector $\tilde{v}$ and

\[
P \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 0 \\ -2/5 \end{bmatrix}.
\]

Then

(A) $P^3 = \frac{1}{5} \left( 7P^2 - 2P \right)$

(B) $P^3 = \frac{1}{4} \left( 7P^2 - 2P \right)$

(C) $P^3 = \frac{1}{3} \left( 7P^2 - 2P \right)$

(D) $P^3 = \frac{1}{2} \left( 7P^2 - 2P \right)$
SECTION - B
MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

Q.31 For two nonzero real numbers $a$ and $b$, consider the system of linear equations

\[
\begin{bmatrix}
    a & b \\
    b & a
\end{bmatrix}
\begin{bmatrix}
    x \\
    y
\end{bmatrix} = \begin{bmatrix}
    b/2 \\
    a/2
\end{bmatrix}.
\]

Which of the following statements is (are) TRUE?

(A) If $a = b$, the solutions of the system lie on the line $x + y = 1/2$
(B) If $a = -b$, the solutions of the system lie on the line $y - x = 1/2$
(C) If $a \neq \pm b$, the system has no solution
(D) If $a \neq \pm b$, the system has a unique solution

Q.32 For $n \geq 1$, let

\[
a_n = \begin{cases} 
  n \cdot 2^{-n}, & \text{if } n \text{ is odd}, \\
  -3^{-n}, & \text{if } n \text{ is even}.
\end{cases}
\]

Which of the following statements is (are) TRUE?

(A) The sequence $\{a_n\}$ converges
(B) The sequence $\left\{\frac{a_n}{1/n}\right\}$ converges
(C) The series $\sum_{n=1}^{\infty} a_n$ converges
(D) The series $\sum_{n=1}^{\infty} |a_n|$ converges

Q.33 Let $f : (0, \infty) \to \mathbb{R}$ be defined by

\[
f(x) = x \left( e^{1/x^3} - 1 + \frac{1}{x^3} \right).
\]

Which of the following statements is (are) TRUE?

(A) $\lim_{x \to \infty} f(x)$ exists
(B) $\lim_{x \to \infty} x f(x)$ exists
(C) $\lim_{x \to \infty} x^2 f(x)$ exists
(D) There exists $m > 0$ such that $\lim_{x \to \infty} x^m f(x)$ does NOT exist.
Q.34 For \( x \in \mathbb{R} \), define \( f(x) = \cos(\pi x) + \left\lfloor x^2 \right\rfloor \) and \( g(x) = \sin(\pi x) \). Which of the following statements is (are) TRUE?

(A) \( f(x) \) is continuous at \( x = 2 \)
(B) \( g(x) \) is continuous at \( x = 2 \)
(C) \( f(x) + g(x) \) is continuous at \( x = 2 \)
(D) \( f(x) g(x) \) is continuous at \( x = 2 \)

Q.35 Let \( E \) and \( F \) be two events with \( 0 < P(E) < 1 \), \( 0 < P(F) < 1 \) and \( P(E \mid F) > P(E) \). Which of the following statements is (are) TRUE?

(A) \( P(F \mid E) > P(F) \)
(B) \( P(E \mid F^c) > P(E) \)
(C) \( P(F \mid E^c) < P(F) \)
(D) \( E \) and \( F \) are independent

Q.36 Let \( X_1, \ldots, X_n \) \((n > 1)\) be a random sample from a \( U(2\theta - 1, 2\theta + 1) \) population, \( \theta \in \mathbb{R} \), and \( Y_i = \min \{X_1, \ldots, X_n\} \), \( Y_n = \max \{X_1, \ldots, X_n\} \). Which of the following statistics is (are) maximum likelihood estimator (s) of \( \theta \)?

(A) \( \frac{1}{4}(Y_i + Y_n) \)
(B) \( \frac{1}{6}(2Y_i + Y_n + 1) \)
(C) \( \frac{1}{8}(Y_i + 3Y_n - 2) \)
(D) Every statistic \( T(X_1, \ldots, X_n) \) satisfying \( \frac{(Y_n - 1)}{2} < T(X_1, \ldots, X_n) < \frac{(Y_i + 1)}{2} \)

Q.37 Let \( X_1, \ldots, X_n \) be a random sample from a \( N(0, \sigma^2) \) population, \( \sigma > 0 \). Which of the following testing problems has (have) the region \( \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \geq \chi^2_{n, \alpha} \right\} \) as the most powerful critical region of level \( \alpha \)?

(A) \( H_0 : \sigma^2 = 1 \) against \( H_1 : \sigma^2 = 2 \)
(B) \( H_0 : \sigma^2 = 1 \) against \( H_1 : \sigma^2 = 4 \)
(C) \( H_0 : \sigma^2 = 2 \) against \( H_1 : \sigma^2 = 1 \)
(D) \( H_0 : \sigma^2 = 1 \) against \( H_1 : \sigma^2 = 0.5 \)
Q.38  Let \( X_1, \ldots, X_n \) be a random sample from a \( N(0, 2\theta^2) \) population, \( \theta > 0 \). Which of the following statements is (are) TRUE?

(A) \( (X_1, \ldots, X_n) \) is sufficient and complete
(B) \( (X_1, \ldots, X_n) \) is sufficient but NOT complete
(C) \( \sum_{i=1}^{n} X_i^2 \) is sufficient and complete
(D) \( \frac{1}{2n} \sum_{i=1}^{n} X_i^2 \) is the uniformly minimum variance unbiased estimator for \( \theta^2 \)

Q.39  Let \( X_1, \ldots, X_n \) be a random sample from a population with the probability density function

\[
f(x | \theta) = \begin{cases} \theta e^{-\theta x}, & x > 0, \\ 0, & \text{otherwise} \end{cases}, \quad \theta > 0.
\]

Which of the following is (are) \( 100(1-\alpha)\% \) confidence interval(s) for \( \theta \)?

(A) \( \left( \frac{X_{2n, \alpha/2}}{2 \sum_{i=1}^{n} X_i}, \frac{X_{2n, \alpha/2}}{2 \sum_{i=1}^{n} X_i} \right) \)

(B) \( \left( 0, \frac{X_{2n, \alpha/2}}{2 \sum_{i=1}^{n} X_i} \right) \)

(C) \( \left( \frac{X_{2n, 1-\alpha/2}}{2 \sum_{i=1}^{n} X_i}, \frac{X_{2n, 1-\alpha/2}}{2 \sum_{i=1}^{n} X_i} \right) \)

(D) \( \left( \frac{2 \sum_{i=1}^{n} X_i}{X_{2n, \alpha/2}}, \frac{2 \sum_{i=1}^{n} X_i}{X_{2n, 1-\alpha/2}} \right) \)

Q.40  The cumulative distribution function of a random variable \( X \) is given by

\[
F(x) = \begin{cases} 0, & x < 2, \\ \frac{1}{10} \left( x^2 - \frac{7}{3} \right), & 2 \leq x < 3, \\ 1, & x \geq 3. 
\end{cases}
\]

Which of the following statements is (are) TRUE?

(A) \( F(x) \) is continuous everywhere
(B) \( F(x) \) increases only by jumps
(C) \( P(X = 2) = \frac{1}{6} \)
(D) \( P\left( X = \frac{5}{2} \mid 2 \leq X \leq 3 \right) = 0 \)
SECTION – C
NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q.41 Let \( X_1, \ldots, X_{10} \) be a random sample from a \( \mathcal{N}(3, 12) \) population. Suppose \( Y_i = \frac{1}{6} \sum_{i=1}^{4} X_i \) and \( Y_2 = \frac{1}{4} \sum_{i=7}^{10} X_i \). If \( \frac{(Y_i - Y_j)^2}{\alpha} \) has a \( \chi^2 \) distribution, then the value of \( \alpha \) is _______________.

Q.42 Let \( X \) be a continuous random variable with the probability density function

\[
f(x) = \begin{cases} 
\frac{2x}{9}, & 0 < x < 3, \\
0, & \text{otherwise.}
\end{cases}
\]

Then the upper bound of \( P(|X - 2| > 1) \) using Chebyshev’s inequality is ________________.

Q.43 Let \( X \) and \( Y \) be continuous random variables with the joint probability density function

\[
f(x, y) = \begin{cases} 
e^{(x+y)}, & -\infty < x, y < 0, \\
0, & \text{otherwise.}
\end{cases}
\]

Then \( P(X < Y) = \) ________________.

Q.44 Let \( X \) and \( Y \) be continuous random variables with the joint probability density function

\[
f(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}, \quad (x, y) \in \mathbb{R}^2.
\]

Then \( P(X > 0, Y < 0) = \) ________________.

Q.45 Let \( Y \) be a \( Bin \left( 72, \frac{1}{3} \right) \) random variable. Using normal approximation to binomial distribution, an approximate value of \( P(22 \leq Y \leq 28) \) is ________________.
Q.46 Let $X$ be a $Bin(2, p)$ random variable and $Y$ be a $Bin(4, p)$ random variable, $0 < p < 1$. If $P(X \geq 1) = \frac{5}{9}$, then $P(Y \geq 1) =$ ____________

Q.47 Consider the linear transformation
\[ T(x, y, z) = (2x + y + z, x + z, 3x + 2y + z). \]
The rank of $T$ is ______________________

Q.48 The value of $\lim_{n \to \infty} n \left[ e^{-n} \cos(4n) + \sin \left( \frac{1}{4n} \right) \right]$ is ______________________

Q.49 Let $f : [0, 13] \to \mathbb{R}$ be defined by $f(x) = x^{13} - e^{-x} + 5x + 6$. The minimum value of the function $f$ on $[0, 13]$ is________________________

Q.50 Consider a differentiable function $f$ on $[0, 1]$ with the derivative $f'(x) = 2\sqrt{2x}$. The arc length of the curve $y = f(x)$, $0 \leq x \leq 1$, is ______________________

**Q. 51 – Q. 60 carry two marks each.**

Q.51 Let $m$ be a real number such that $m > 1$. If
\[ \int_0^1 \int_0^{\sqrt[3]{m}} e^{y^3} \, dy \, dx = e - 1, \]
then $m =$ ______________________

Q.52 Let
\[ P = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}. \]
The product of the eigenvalues of $P^{-1}$ is ______________________

Q.53 The value of the real number $m$ in the following equation
\[
\int_0^1 \int_x^1 \sqrt{2-x^2} \, (x^2+y^2) \, dy \, dx = \int_0^{\pi/2} \int_0^r r^3 \, dr \, d\theta
\]
is ____________________

Q.54 Let $a_1 = 1$ and $a_n = 2 - \frac{1}{n}$ for $n \geq 2$. Then
\[
\sum_{n=1}^{\infty} \left( \frac{1}{a_n^2} - \frac{1}{a_{n+1}^2} \right)
\]
converges to ____________________

Q.55 Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with the probability density function
\[
f(x) = \begin{cases} 
4x^2 e^{-2x}, & x > 0, \\
0, & \text{otherwise}
\end{cases}
\]
and let $S_n = \sum_{i=1}^{n} X_i$. Then $\lim_{n \to \infty} P \left( S_n \leq \frac{3n}{2} + \sqrt{3n} \right)$ is ____________________

Q.56 Let $X$ and $Y$ be continuous random variables with the joint probability density function
\[
f(x, y) = \begin{cases} 
c \frac{x^2}{y^3}, & 0 < x < 1, \, y > 1, \\
0, & \text{otherwise}
\end{cases}
\]
where $c$ is a suitable constant. Then $E(X) =$ ____________________

Q.57 Two points are chosen at random on a line segment of length 9 cm. The probability that the distance between these two points is less than 3 cm is ____________________
Q.58 Let $X$ be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then $P\left(\frac{1}{4} < X^2 < \frac{1}{2}\right) =$ ____________________

Q.59 If $X$ is a $U(0,1)$ random variable, then $P\left(\min(X, 1-X) \leq \frac{1}{4}\right) =$ ____________________

Q.60 In a colony all families have at least one child. The probability that a randomly chosen family from this colony has exactly $k$ children is $\left(0.5\right)^k$; $k = 1, 2, \ldots$. A child is either a male or a female with equal probability. The probability that such a family consists of at least one male child and at least one female child is ________

END OF THE QUESTION PAPER