SECTION 1

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : $+4$ If only (all) the correct option(s) is(are) chosen;
  Partial Marks : $+3$ If all the four options are correct but ONLY three options are chosen;
  Partial Marks : $+2$ If three or more options are correct but ONLY two options are chosen, both of which are correct;
  Partial Marks : $+1$ If two or more options are correct but ONLY one option is chosen and it is a correct option;
  Zero Marks : 0 If unanswered;
  Negative Marks : -2 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  choosing ONLY (A), (B) and (D) will get $+4$ marks;
  choosing ONLY (A) and (B) will get $+2$ marks;
  choosing ONLY (A) and (D) will get $+2$marks;
  choosing ONLY (B) and (D) will get $+2$ marks;
  choosing ONLY (A) will get $+1$ mark;
  choosing ONLY (B) will get $+1$ mark;
  choosing ONLY (D) will get $+1$ mark;
  choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
  choosing any other option(s) will get -2 marks.

Q.1 Let

$$S_1 = \{(i,j,k) : i,j,k \in \{1,2,\dots,10\}\},$$
$$S_2 = \{(i,j) : 1 \le i < j+2 \le 10,\ i,j \in \{1,2,\dots,10\}\},$$
$$S_3 = \{(i,j,k,l) : 1 \le i < j < k < l,\ i,j,k,l \in \{1,2,\dots,10\}\}$$

and

$$S_4 = \{(i,j,k,l) : i,j,k \text{ and } l \text{ are distinct elements in } \{1,2,\dots,10\}\}.$$

If the total number of elements in the set S_r is n_r, $r = 1,2,3,4$, then which of the following statements is (are) **TRUE** ?

(A) $n_1 = 1000$ (B) $n_2 = 44$ (C) $n_3 = 220$ (D) $\frac{n_4}{12} = 420$

Q.2 Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) TRUE ?

(A) $\cos P \geq 1 - \frac{p^2}{2qr}$

(B) $\cos R \geq \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)\cos Q$

(C) $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$

(D) If $p < q$ and $p < r$, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

Q.3 Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ be a continuous function such that

$$f(0) = 1 \text{ and } \int_0^{\frac{\pi}{3}} f(t)\,dt = 0$$

Then which of the following statements is (are) TRUE ?

(A) The equation $f(x) - 3\cos 3x = 0$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$

(B) The equation $f(x) - 3\sin 3x = -\frac{6}{\pi}$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$

(C) $\lim_{x \to 0} \frac{x\int_0^x f(t)\,dt}{1 - e^{x^2}} = -1$

(D) $\lim_{x \to 0} \frac{\sin x \int_0^x f(t)\,dt}{x^2} = -1$

Q.4 For any real numbers α and β, let $y_{\alpha,\beta}(x), x \in \mathbb{R}$, be the solution of the differential equation

$$\frac{dy}{dx} + \alpha y = x e^{\beta x}, \quad y(1) = 1.$$

Let $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S?

(A) $f(x) = \frac{x^2}{2} e^{-x} + \left(e - \frac{1}{2}\right) e^{-x}$

(B) $f(x) = -\frac{x^2}{2} e^{-x} + \left(e + \frac{1}{2}\right) e^{-x}$

(C) $f(x) = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right) e^{-x}$

(D) $f(x) = \frac{e^x}{2}\left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right) e^{-x}$

Q.5 Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}\left(\overrightarrow{OB} - \lambda \overrightarrow{OA}\right)$ for some $\lambda > 0$. If $\left|\overrightarrow{OB} \times \overrightarrow{OC}\right| = \frac{9}{2}$, then which of the following statements is (are) TRUE ?

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

Q.6 Let E denote the parabola $y^2 = 8x$. Let $P = (-2, 4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) TRUE ?

(A) The triangle PFQ is a right-angled triangle

(B) The triangle QPQ' is a right-angled triangle

(C) The distance between P and F is $5\sqrt{2}$

(D) F lies on the line joining Q and Q'

SECTION 2

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, truncate/round-off the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If ONLY the correct numerical value is entered in the designated place;
Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 7 and 8

Question Stem

Consider the region $R=\{(x,y)\in\mathbb{R}\times\mathbb{R} : x\ge 0 \text{ and } y^2\le 4-x\}$. Let $\mathcal{F}$ be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in $\mathcal{F}$. Let (α,β) be a point where the circle C meets the curve $y^2=4-x$.

Q.7 The radius of the circle C is ___.

Q.8 The value of α is ___ .

Question Stem for Question Nos. 9 and 10

Question Stem

Let $f_1:(0,\infty)\to\mathbb{R}$ and $f_2:(0,\infty)\to\mathbb{R}$ be defined by

$$f_1(x)=\int_0^x \prod_{j=1}^{21}(t-j)^j\,dt,\quad x>0$$

and

$$f_2(x)=98(x-1)^{50}-600(x-1)^{49}+2450,\quad x>0,$$

where, for any positive integer n and real numbers $a_1, a_2, \ldots, a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1, a_2, \ldots, a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i=1,2$, in the interval $(0,\infty)$.

Q.9 The value of $2m_1+3n_1+m_1n_1$ is ____.

Q.10 The value of $6m_2+4n_2+8m_2n_2$ is ____.

Question Stem for Question Nos. 11 and 12

Question Stem

Let $g_i:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$, $i=1,2$, and $f:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$ be functions such that

$$g_1(x)=1,\quad g_2(x)=|4x-\pi| \text{ and } f(x)=\sin^2 x, \text{ for all } x\in\left[\frac{\pi}{8},\frac{3\pi}{8}\right]$$

Define

$$S_i=\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x)\cdot g_i(x)\,dx,\quad i=1,2$$

Q.11 The value of $\frac{16S_1}{\pi}$ is ____.

Q.12 The value of $\frac{48S_2}{\pi^2}$ is ____.

SECTION 3

- This section contains **TWO (02)** paragraphs. Based on each paragraph, there are **TWO (02)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : $+3$ If ONLY the correct option is chosen;
  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
  Negative Marks : -1 In all other cases.

Paragraph

Let

$$M = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \le r^2\},$$

where $r > 0$. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}$, $n = 1, 2, 3, \ldots$. Let $S_0 = 0$ and, for $n \ge 1$, let S_n denote the sum of the first n terms of this progression. For $n \ge 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n, and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n.

Q.13 Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then

(A) $k + 2l = 22$ (B) $2k + l = 26$ (C) $2k + 3l = 34$ (D) $3k + 2l = 40$

Q.14 Consider M with $r = \frac{(2^{199} - 1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

(A) 198 (B) 199 (C) 200 (D) 201

Paragraph

Let $\psi_1 : [0, \infty) \to \mathbb{R}$, $\psi_2 : [0, \infty) \to \mathbb{R}$, $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be functions such that $f(0) = g(0) = 0$,

$$\psi_1(x) = e^{-x} + x, \quad x \geq 0,$$
$$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2, \quad x \geq 0,$$
$$f(x) = \int_{-x}^{x} \left(|t| - t^2\right) e^{-t^2}\, dt, \quad x > 0$$

and

$$g(x) = \int_{0}^{x^2} \sqrt{t}\, e^{-t}\, dt, \quad x > 0.$$

Q.15 Which of the following statements is TRUE ?

(A) $f\left(\sqrt{\ln 3}\right) + g\left(\sqrt{\ln 3}\right) = \frac{1}{3}$

(B) For every $x > 1$, there exists an $\alpha \in (1, x)$ such that $\psi_1(x) = 1 + \alpha x$

(C) For every $x > 0$, there exists a $\beta \in (0, x)$ such that $\psi_2(x) = 2x(\psi_1(\beta) - 1)$

(D) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

Q.16 Which of the following statements is TRUE ?

(A) $\psi_1(x) \leq 1$, for all $x > 0$

(B) $\psi_2(x) \leq 0$, for all $x > 0$

(C) $f(x) \geq 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \leq \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

SECTION 4

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : $+4$ If ONLY the correct integer is entered;

  Zero Marks : 0 In all other cases.

Q.17 A number is chosen at random from the set $\{1, 2, 3, \ldots, 2000\}$. Let p be the probability that the chosen number is a multiple of 3 or a multiple of 7. Then the value of $500p$ is _____.

Q.18 Let E be the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. For any three distinct points P, Q and Q' on E, let $M(P, Q)$ be the mid-point of the line segment joining P and Q, and $M(P, Q')$ be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between $M(P, Q)$ and $M(P, Q')$, as P, Q and Q' vary on E, is _____.

Q.19 For any real number x, let $[x]$ denote the largest integer less than or equal to x. If

$$I = \int_0^{10} \left[\sqrt{\frac{10x}{x+1}}\right] dx,$$

then the value of $9I$ is _____.

END OF THE QUESTION PAPER